Self-Descriptive Number

L. Stabile
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A self-descriptive number in base 10 is a 10-digit number x, the digits of x being numbered from most to least significant as $d_0, \ldots, d_9$, such that d_i represents the quantity of the digit i in the decimal representation of x. [1] We start by finding a digit of x. This is simply

$$d_i(x) = [\lfloor \frac{x}{10^{9-i}} \rfloor]_{10}$$

where the notation $[x]_m$ denotes the common residue of x *mod* m. Next, recall Fermat's Little Theorem, which states that, for any natural number a and any prime p

$$a^p \equiv a \ (mod\ p)$$

When p does not divide a,

$$a^{p-1} \equiv 1 \ (mod\ p)$$

and under these conditions the mod operator produces a function which acts as a zero detector:

$$[a^{p-1}]_p = \begin{cases} 1, & a \neq 0 \\ 0, & a = 0 \end{cases}$$

for $0 \leq a < p$.

Suppose therefore we wish to count the number of zero digits in the decimal representation of x. For the decimal-oriented problem at hand, we select for p the smallest prime greater than the largest digit involved, in this case 11, and define

$$z_0(x) = \sum_{i=0}^{9} 1 - [(d_i(x))^{10}]_{11}$$

To generalize this to any digit, we normalize a given digit n to zero, i.e., $[n - d_i(x)]_{10}$, so that

[1] See Weisstein, Eric W. "Self-Descriptive Number." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/Self-DescriptiveNumber.html

$$z_n(x) = \sum_{i=0}^{9} 1 - [([n - d_i(x)]_{10})^{10}]_{11}$$

$z_n(x)$ gives us the digit which is the number of digits of value n in x. The value across all digits is thus

$$z(x) = \sum_{n=0}^{9} z_n(x) 10^{9-n}$$

When expanded, this becomes

$$z(x) = \sum_{n=0}^{9} (\sum_{i=0}^{9} 1 - [([n - \lfloor \frac{x}{10^{9-i}} \rfloor]_{10})^{10}]_{11}) 10^{9-n}$$

It is readily verified that 6210001000 is a fixed point of $z(x)$.